Indian Statistical Institute
Semestral Examination
Differential Topology: MMath II

Max Marks: 60 Time: 3 hours

(1) (a) Does there exist a smooth one-one map $f : S^2 \longrightarrow S^1$? Justify. [4]

(b) Identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by identifying $(z_1, z_2) \in \mathbb{C}^2$ with $(x, y, u, v) \in \mathbb{R}^4$ where $z_1 = x + iy$ and $z_2 = u + iv$. Let

$$X = \{(z_1, z_2) : z_1^3 + z_2^2 = 0\}.$$

Show that $X - \{(0, 0)\}$ is a manifold. Find its dimension. [6]

(c) For which values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally. [6]

(d) Let X be a manifold. Prove that there exists a proper map $f : X \longrightarrow \mathbb{R}$. [4]

(2) (a) Let $f : \mathbb{R}^2 - 0 \longrightarrow \mathbb{R}^2$ be the map defined by

$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

Let Z denote the unit circle. Show that f is not transverse to Z. Construct a map g such that g is homotopic to f and transverse to Z. [6]

(b) Discuss the definition of the intersection number $I_2(f, Z)$ where $f : X \longrightarrow Y$ is a map and Z a submanifold of Y. State your assumptions clearly. Show that if $f_0, f_1 : X \longrightarrow Y$ are homotopic, Z a submanifold of Y, and $I_2(f_0, Z)$ is defined, then $I_2(f_0, Z) = I_2(f_1, Z)$. [8]

(c) Prove that every $n \times n$ real matrix A with nonnegative entries has a real nonnegative eigenvalue. [6]

(3) (a) Construct a nowhere zero n-form on S^n. [5]

(b) Let ω be an exact 1-form on S^1. Show that $\omega(x) = 0$ for some $x \in S^1$. Is the same conclusion true for exact 1-forms on $\mathbb{R}$ and $[0, 1]$? [6]

(c) Show that $H^1(\mathbb{R}^2 - 0) \neq 0$. [9]